Senior Mathematical Challenge

Tuesday 1 October 2024

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1. **E** Written as a fraction, $0.\dot{2}\dot{5}$ is $\frac{25}{99}$. One-fifth of the number is $\frac{5}{99}$ so two-fifths = $\frac{10}{99} = 0.\dot{1}\dot{0}$.

2. **A** The number of ***twips*** in a ***league*** is $\dfrac{4800\,\text{m}}{0.000018\,\text{m}} = \dfrac{4.8 \times 10^3}{1.8 \times 10^{-5}} = \dfrac{8}{3} \times 10^8 = 2.\dot{6} \times 10^8 \approx 270000000$.

3. **B** The numbers on opposite faces of a standard dice sum to 7. On the bottom dice, there are two pairs of opposite faces which are visible. On the top dice, there are again two pairs of opposite faces which are visible along with the number on the top of that dice, n say. Therefore, $(2 + 2) \times 7 + n = 33$ and so $n = 5$. The number on the touching faces is then $7 - 5 = 2$.

4. **C** As the angle sum of a triangle is 180°, $x + 7x + x^2 = 180$. Therefore, $x^2 + 8x - 180 = 0$ which factorises to $(x + 18)(x - 10) = 0$. As $x > 0$, $x = 10$. The angles are then 10°, 70° and 100°, so the largest angle is 100°.

5. **B** $4^5 \times 5^4 = 2^{10} \times 5^4 = 2^6 \times 2^4 \times 5^4 = 2^6 \times 10^4 = 640000$. Hence the answer has six digits.

6. **D** The solid with the minimum number of edges must contain an octagonal ‘base’. This has 8 edges and 8 vertices. Including one extra edge from each of these vertices, all of which meet at a single vertex that is not on the base, creates an octagonal pyramid. There are $8 + 8 = 16$ edges.

7. **A** In order to write the expression $3^8 - 1$ as the product of its primes without first calculating its value, we can use the difference of two squares: $3^8 - 1 = (3^4 + 1)(3^4 - 1) = 82 \times 80 = 41 \times 2 \times 2^4 \times 5$. Hence the largest prime factor is 41.

8. **D** The mean of four terms is $11x$ so the sum of those four terms is $44x$. As the total of the five options is $4x + 8x + 12x + 16x + 20x = 60x$, the term to exclude from the sum is $60x - 44x = 16x$.

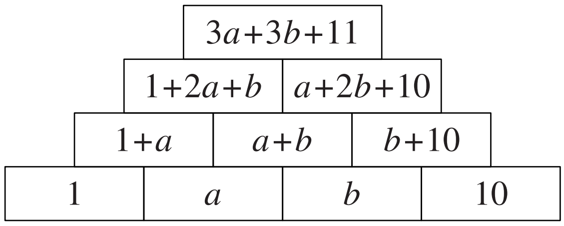
9. **E** For the number we seek to be divisible by 18, it must be divisible by both 2 and 9. To be even, it must have a last digit of 0, 2, 4, 6 or 8. This last digit is also the first digit of the palindromic number, so we choose it to be an 8. To be divisible by 9, the sum of all six digits must be a multiple of 9. As we already have two 8s, the remaining four middle digits, with a maximum sum of $4 \times 9 = 36$, could sum to 2, 11, 20 or 29. The sum of these four digits must itself be even as there are two repeated pairs. So we require the second and third numbers from the left to sum to $\frac{20}{2} = 10$ at the same time as maximising the second digit due to its place value. Hence the number we seek is 891198 and the hundreds digit is a 1.

10. **D** As $2024 = 2 \times 2 \times 2 \times 11 \times 23$, the two-digit numbers which are factors of 2024 are either suitable multiples of 11: 11, $11 \times 2 = 22$, $11 \times 2 \times 2 = 44$, $11 \times 2 \times 2 \times 2 = 88$ or multiples of 23: 23, $23 \times 2 = 46$, $23 \times 2 \times 2 = 92$. In total, this gives seven two-digit factors.

11. **D** Choosing n to be 1, say, eliminates options A, B and C as the resulting values 2, 3 and 7 are not square. Similarly, choosing n to be 2 eliminates option E as 721 is not square. Now considering option D, we can see that $n(n+1)(n+2)(n+3)+1 = n^4 + 6n^3 + 11n^2 + 6n + 1$ which can be written as $(n^2 + 3n + 1)^2$ and is therefore a square for each positive integer n.

12. **A** In order to create two-digit primes, the units digits must be 1, 3, 7 and 9 in some order. Therefore the tens digits must be 2, 4, 6 and 8 in some order. We can add together all the tens and all the units without being concerned which combines with which to create the actual primes, so $p + q + r + s = (20 + 40 + 60 + 80) + (1 + 3 + 7 + 9) = 220$. There are in fact four possible ways to assign the digits to create p, q, r and s.

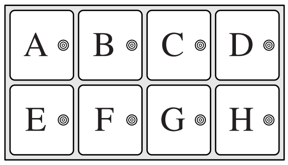
13. **C** Labelling the missing numbers in the bottom row as a and b, the pyramid can be filled as shown. Therefore $3a + 3b + 11 = 2024$, and so subtracting 11 and dividing by 3 gives $a + b = 671$. Hence 671 appears on the brick marked z.



14. **B** In order to be primes, the three-digit numbers cannot end in 2, 4 or 5. S and T must therefore be 1 and 3 in either order. We may assume that $S = 3$ so that $T = 1$. Now Q and R cannot be 2 and 4 in either order nor 4 and 5 in either order as then QRS would be a multiple of 3. Therefore Q and R must be 2 and 5 in some order. However $253 = 23 \times 11$. So $QRS = 523$ which is (and in the context of the SMC, must be) a prime. Therefore $R = 2$ and then $PRT = 421$ which is also a prime.

15. **C** Let $KL = x$ cm and $KN = y$ cm. As the shaded area is 62, $x^2 - y^2 = 62$. Applying Pythagoras' Theorem to triangle NKL gives $x^2 + y^2 = 10^2$. Subtracting the first equation from the second gives $2y^2 = 38$ so $y^2 = 19$ and hence $y = \sqrt{19}$.

16. **B** Let the doors be labelled A to H as shown. If all three red doors are corners, the one which remains blue can be A, D, E or H. This gives 4 ways. Now suppose that just two corners are red. If those two corners are in different rows, (that is AE, AH, DE or DH) then the third red door is any one of B, C, F or G giving four ways for each of the four cases. This gives another 16 ways.



Finally, suppose the two red corner doors are in the same row. If they are AD then the third door must be F or G and if they are EH the third must be B or C. This gives a further 4 ways. So the total number of ways is $4 + 16 + 4 = 24$.

17. **E** We start by finding how many red balls are in the bag. Let this number be n. Therefore the probability of two reds is $\frac{n}{4} \times \frac{(n-1)}{3} = \frac{1}{2}$. This rearranges to give $n^2 - n - 6 = 0$, which factorises to $(n-3)(n+2) = 0$. As $n \geq 0$, $n = 3$. So there are three red balls and only one white. Hence the probability that both balls are white is 0.

18. **C** The area of each of the four central quadrants is $\frac{1}{4} \times \pi \times 1^2 = \frac{\pi}{4}$. Therefore the area enclosed by the outer circle is $\frac{14\pi}{4}$. The radius of the outer circle is $1 + x$, therefore $\pi \times (1+x)^2 = \frac{14\pi}{4}$, which rearranges to $2x^2 + 4x - 5 = 0$. As $x > 0$, the quadratic formula leads us to $x = \frac{\sqrt{14}}{2} - 1$.

19. C In order to find the correct option, we will try to determine which cards are held by which friend. Here is a list of pairs of cards that are feasible for each friend, given their declared totals.

Pablo (4)	Quinn (11)	Romy (16)	Stephen (19)	Thomas (20)
3, 1	10, 1	12, 4	12, 7	12, 8
	9, 2	11, 5	11, 8	11, 9
	8, 3	10, 6	10, 9	
	7, 4	9, 7		
	6, 5			

The total of all the cards 1 to 12 is 78. The cards held by the friends sum to $4+11+16+19+20 = 70$ so the unused cards sum to 8. Paolo's total is 4 so he has 1 and 3. The unused cards must then be 2 and 6.The only possibilities for Thomas to have 20 are 12, 8 or 11, 9. Suppose that Thomas has 11, 9. Then Stephen must have 12, 7. However, then there is no way for Romy to have 16. Hence Thomas must have 12, 8. Then Stephen has 10, 9, Romy has 11, 5 and Quinn has 7, 4.

Pablo (4)	Quinn (11)	Romy (16)	Stephen (19)	Thomas (20)
3, 1	7, 4	11, 5	10, 9	12, 8

Of the options given, only *C* is true.

20. D Rearranging $\frac{1}{x} + \frac{1}{y} = \frac{1}{20}$ gives $y = \frac{20x}{x-20} = 20 + \frac{400}{x-20}$. In order to maximise y, we require $x - 20$ to be as small as possible. As x is an integer, $x = 21$. Then $y = 20 + \frac{400}{21-20} = 420$.

21. B We begin where we have least choice. Here that is '2 Down'. The smallest multiple of both 13 and 19 is $13 \times 19 = 247$. The list of all such three-digit multiples is 247, 494, 741 and 988. As digits may not be repeated we have only 247 and 741. The units digit of '2 Down' is also the units digit of '3 Across', 'A square'. As squares do not end in 7, '2 Down' must be 741. Considering '3 Across', three-digit squares which end in 1 come from $11^2, 21^2, 31^2, 19^2$ or 29^2. However, without repeated digits or use of 4 or 7, our only possibilities are $31^2 = 961$ or $19^2 = 361$. Now considering '1 Down' we look for multiples of 11 which end in either 3 or 9. Multiples ending in 3 come from the answers to $11 \times 13, 11 \times 23, \ldots, 11 \times 83$. Given the remaining available digits, this is either $11 \times 23 = 253$ or $11 \times 53 = 583$. Multiples ending in 9 come from the answers to $11 \times 19, 11 \times 29, \ldots, 11 \times 89$. Given the remaining available digits, this can only be $11 \times 49 = 539$. At this stage we have three cases under consideration.

To complete the crossnumber with a multiple of 9 in '1 Across', we require the digits in the top row to sum to a multiple of 9. If '1 Down' were to be either 539 or 583 the middle digit would need to be a 6 so that $5+6+7 = 18$. However the 6 has already been used. '1 Down' must therefore be the only remaining possibility, 253, and '1 Across' must be 297.The completed grid is as shown. The digit which is not used is 8.

(1) 2	9	(2) 7
5		4
(3) 3	6	1

22. A

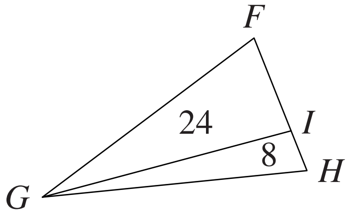


Fig. 1

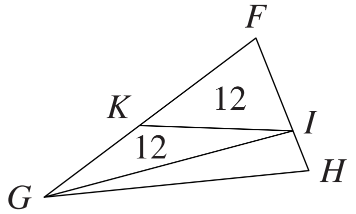


Fig. 2

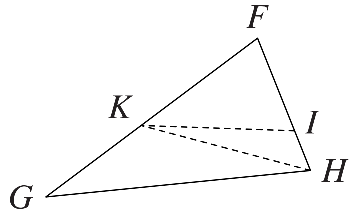
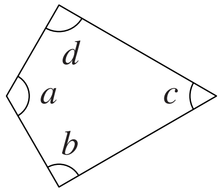


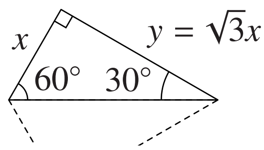
Fig. 3

Using 'area of a triangle = $\frac{1}{2}$ base $\times$ perpendicular height' with FH as the base, triangles FGI and FGH have the same perpendicular height. Their areas are therefore in the same proportions as the lengths of their bases and so $IH = \frac{1}{4}FH$.
Now viewing FI as the base of both triangles FGI and FKI, we can deduce that the perpendicular height from FI to K is half the perpendicular height from FI to G.
The area of triangle $IKH = \frac{1}{2}\ IH \times$ the perpendicular height from IH to $K = \frac{1}{2} \times \frac{1}{4}FH \times \frac{1}{2}$ the perpendicular distance from IH to $G = \frac{1}{8} \times$ the area of triangle $FGH = \frac{1}{8} \times 32 = 4$.

23. E On the diagram shown in the question, we can see that each kite has a line of symmetry. Therefore $\angle b = \angle d$. Also, where three kites meet at a point with no gaps, $\angle a = 120°$. Where four kites meet at a point with no gaps, $\angle b = \angle d = 90°$. As the angle sum of a quadrilateral is $360°$, $\angle c = 60°$. Each half-kite is therefore a $30°, 60°, 90°$ triangle, with lengths in the ratio $1 : \sqrt{3} : 2$.

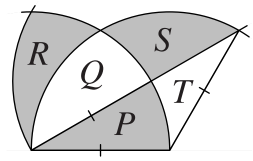


Let the perpendicular lengths be x and y as shown. So $y = \sqrt{3}x$. As the area of the whole hat tile is $8\sqrt{3}$, the area of each kite is $\sqrt{3}$. Therefore $2 \times \frac{xy}{2} = \sqrt{3}$, so $\sqrt{3}x^2 = \sqrt{3}$ and $x = 1$. Therefore $y = \sqrt{3}$. The perimeter of the hat tile is $8x + 6y = 8 \times 1 + 6 \times \sqrt{3} = 8 + 6\sqrt{3}$.

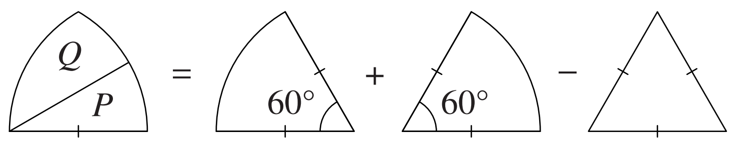


24. E First let $x = 3$, then $f(3) + f(\frac{1}{1-3}) = 24 \times 3$. Therefore $f(3) + f(-\frac{1}{2}) = 72$ (a). Now let $x = -\frac{1}{2}$, then $f(-\frac{1}{2}) + f(\frac{1}{1-\frac{-1}{2}}) = 24 \times -\frac{1}{2}$. So $f(-\frac{1}{2}) + f(\frac{2}{3}) = -12$ and thus $-f(-\frac{1}{2}) - f(\frac{2}{3}) = 12$ (b). Finally, let $x = \frac{2}{3}$, then $f(\frac{2}{3}) + f(\frac{1}{1-\frac{2}{3}}) = 24 \times \frac{2}{3}$. This simplifies to $f(\frac{2}{3}) + f(3) = 16$ (c). Adding equations (a), (b) and (c) leads to $2 \times f(3) = 72 + 12 + 16$. Therefore $f(3) = 50$.

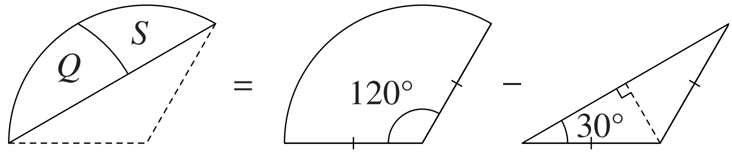
25. A The diagram shows regions P, Q, R, S and T. As $\angle ZXY = 30°$, area of $P = \frac{1}{6}$ of the area of a semicircle $= \frac{1}{6} \times 24 = 4$. Regions $(P + Q)$ and $(Q + R)$ are congruent therefore area of $(P + Q)$ = area of $(Q + R)$ and so area of R = area of $P = 4$. We can also show that the area of $(P + Q)$ is the same as that of $(Q + S)$ as follows:



Region $(P + Q)$ can be deconstructed into two overlapping, congruent $60°$ sectors minus an equilateral triangle.



Region $(Q + S)$ can be deconstructed into a $120°$ sector, minus two $30°, 60°, 90°$ triangles whose total area is that of the equilateral triangle shown in the deconstruction of the region $(P + Q)$.



Therefore area of $(P + Q)$ = area of $(Q + S)$ and so area of P = area of $S = 4$. Hence the total area of the shaded region $= 4 + 4 + 4 = 12$.